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A New Upper Bound for the Capacity of Free Space Optical Intensity Channel by Using a Simple Mathematical Inequality

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Abstract
Farid-Hranilovic (FH), in an interesting way, found a capacity-achieving discrete input distribution for free space optical (FSO) channel by numerically maximizing the input-parameter (β) dependent mutual information between channel input and the scaled output. In this paper, first, by using a simple mathematical inequality, we find an upper bound for FH input-scaled output mutual information and then maximize the obtained upper bound to reach to a third order equation for the optimum β as β^*. Our equation (i) determines β^* exactly in contrary to the FH work where β^* is found numerically through an exhaustive search and also, (ii) is consistent with the estimated equation for β^* in the FH work. Our upper bound is shown to be tighter than the proposed upper bound in the FH work that is found through sphere packing argument at very high SNRs. Using numerical illustrations at different SNRs, we compare our β^*s, mass point spacing as ℓ^*, and upper bound with previous works.

Keywords: *FSO channels, equally spaced mass points, maximization of the mutual information.*

1. Introduction

Free space optical (FSO) channels provide an economical high-speed link for wireless access. They also provide spectral segment and high security. Recently the use of FSO channels and study on them is being specially interesting. They are used in a lot of urban areas, as well as a supplement to radio frequency (RF) links and the recent development of radio. A majority of wireless optical channels are intensity modulated with direct detection [1]. The receiver usually consists of a photo-detector that measures the optical intensity of the incoming light and produces an output signal which is proportional to the detected intensity, corrupted by noise. In these channels data are transmitted by modulating the instantaneous intensity of a laser or a LED. So all the transmitted signals are non-negative. In addition, an average optical power, i.e., average amplitude, constraint is imposed on the transmitted signal, due to eye safety and physical limitations [1,2]. A peak amplitude constraint is also applied due to safety. A good discrete time representation for this channel is given by [1],

$$Y = rX+Z, \qquad (1)$$

where X, is the transmitted signal, Y, is the output signal and Z, models both thermal noise and ambient light induced shot noise and can be well modelled as zero mean, signal independent, Gaussian noise with variance σ^2 or $Z \sim \mathrm{N}(0, \sigma^2)$. The constant r represents channel coherent fading and without loss of generality, we set r = 1. Consraints are:

$$X \geq 0, \quad E\{X\} \leq P, \qquad (2)$$

where P is the average power limit. Notice that although the transmitted signal, X must be nonnegative, the output signal, Y can be either negative or positive. The optical SNR is defined as P/σ. Channel capacity is defined as the maximum mutual information between the channel input and output under nonnegativity and average optical power constraint where the maximization is carried out over all possible input distributions.

Previous Works

It was shown in [3] that the capacity-achieving input distributions for channels with constrained input amplitude and power are discrete with a finit number of mass points.

Similar results were obtained for optical photon-counting, i.e., Poisson channels, with optical power constraints [4].

In other words, the input distribution that maximizes the mutual information is defined as,

$Q = \{q_X(x): q_X(x) = \sum_{k=0}^{m} a_k\,\delta(x - x_k), a_k \geq 0,\ x_k \geq 0,\ \sum_{k=0}^{m} a_k = 1,\ m \in Z^+,\ P \geq \sum_{k=0}^{m} a_k x_k\}$, (3)

where $\delta(.)$ is the delta function, a_k and x_k are the amplitude and position of the kth mass point respectively, and Z^+ is the set of positive integers. The number of mass points is $m+1$. Finding this distribution requires solving a complex non-linear optimization at each SNR.

In [5], FH presented a family of discrete time distribution with equally spaced mass points derived via source entropy maximization when $r = 1$. A capacity-approaching input distribution was obtained numerically by maximizing the mutual information over the maxentropic input distribution.

In [6], FH generalized their work in [5], channel fading was also considered and CSI was available at both the transmitter and the receiver (coherent). They modelled channel loss and fading by a Gamma-Gamma distribution. A capacity-approaching input distribution was obtained numerically by maximizing the mutual information over the maxentropic input distribution.

In [7], a new closed-form upper bound on the capacity of power constrained optical wireless links was derived when on-off keying (OOK) formats were used. CSI was available at both the transmitter and the receiver. Channel capacity was considered as a random variable following the Gamma-Gamma distribution corresponding to the atmospheric turbulence model. A maximization was carried out on the average value of the channel capacity known as the ergodic capacity over the input distribution and the upper bound was computed.

In [8], a general upper bound for the mutual information of FSO channels was found when CSI was available at both the transmitter and the receiver (coherent) through a simple mathematical inequality, and a closed-form for the corresponding optimal input distribution was obtained by maximizing the mutual information over all the discrete input distributions with equally spaced mass points. The approximated optimal input distributions were found directly through a second order equation that was independent of channel parameters, in contrary to [5,6,7], where the optimal input distributions were found numerically.

In [1,2], FH first considered a family of maxentropic input distribution with equally spaced mass points subject to the constraints,

$q_X(x) = \sum_{k=0}^{m} a_k\,\delta(x - k\ell)$, (4)

$q^*_X(x) \triangleq \arg\max H(x),\ q_x(x)$ (5)

$H(x) = -\sum_x q_X(x)\log_2 q_X(x) = -\sum_{k=0}^{m} a_k \log_2 a_k$, (6)

where ℓ is the mass point spacing and $q^*_x(x)$ is the optimal input distribution that maximizes source entropy. The constraints are:

$\sum_{k=0}^{m} a_k = 1\ ,\ \ P = \sum_{k=0}^{m} k\ell a_k.$

Note that although any distribution for $q_X(x)$ was sufficient to provide a lower bound, FH proposed selecting the maxentropic distribution subject to the constraints under the intuition that it was close to the capacity at high SNRs.

Applying the method of lagrange multipliers, the optimal parametrized input distribution was given by,

$$q^*_X(x) = \sum_{k=0}^{m} \frac{\ell}{\ell+P}\left(\frac{P}{\ell+P}\right)^k \delta(x - k\ell), \quad (7)$$

second they scaled the output to,

$$W = Y/\ell = \frac{X+Z}{\ell} = \frac{X}{\ell} + Z_\ell, \quad (8)$$

$$Z \sim (0,\sigma^2) \rightarrow Z_\ell = \frac{Z}{\ell} \sim (0, \beta^2 = \frac{\sigma^2}{\ell^2}),$$

where Z_ℓ is the scaled output noise with zero mean and variance β^2.

Reparametrizing the parametrized optimal input distribution in (7) with β,

$$q^*_X(x) = \sum_{k=0}^{m} \frac{1}{1+\beta\frac{P}{\sigma}}\left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^k \delta(x - k\ell). \quad (9)$$

The scaled output distribution remained Gaussian for every k and had the PDF below,

$$f_W(w) = \sum_{k=0}^{m} \frac{1}{1+\beta\frac{P}{s}}\left(\frac{\beta\frac{P}{s}}{1+\beta\frac{P}{s}}\right)^k \frac{1}{\sqrt{2\pi\beta^2}}\, e^{-\frac{(w-k)^2}{2\beta^2}} \quad (10)$$

Then they found a lower bound for the capacity of FSO channel for each SNR as $I_{q,\beta}(X;W)$,

$$I_{q,\beta}(X;Y) = I_{q,\beta}(X;W) = H(W) - H(W|X) = H(W) - H(Z_\ell) = -\int f_W(w)\log_2 f_W(w)\,dw - \frac{1}{2}\log_2(2\pi e\beta^2)$$

$$= -\int_{-\infty}^{\infty} \sum_{k=0}^{m} \frac{1}{1+\beta\frac{P}{s}}\left(\frac{\beta\frac{P}{s}}{1+\beta\frac{P}{s}}\right)^k \frac{1}{\sqrt{2\pi\beta^2}}\, e^{-\frac{(w-k)^2}{2\beta^2}}.$$

$$\log_2 \sum_{i=0}^{m} \frac{1}{1+\beta\frac{P}{s}} \left(\frac{\beta\frac{P}{s}}{1+\beta\frac{P}{s}}\right)^i \frac{1}{\sqrt{2\pi\beta^2}}\, e^{-\frac{(w-i)^2}{2\beta^2}}\, dw - \frac{1}{2}\log_2(2\pi e\beta^2). \quad (11)$$

In [2], for a given P/σ maximization of the mutual information given in (11) as,

$$\beta^*(\mathrm{SNR}) = \arg\max_{q,\beta} I_{q,\beta}(X;W), \quad (12)$$

was solved numerically using the bisection method over wide range of β to find the lower bound, and no analytic form was provided. The lower bound was computed through numerical integration.

In [1], maximization of the mutual information in (11) was carried out by discretizing the range $\beta \in (0, 1]$ with an increment of $\Delta\beta = 5 \times 10^{-4}$. For each entry in the discrete set, the mutual information was computed numerically to see if it became maximum or not.

A new upper bound for the capacity of FSO channels was also obtained through a sphere packing argument in [1,2].

Our Work

By using a simple mathematical inequality we find a new upper bound for the FH input-scaled output mutual information. Then, by maximizing the obtained upper bound over all βs, we reach to a third order equation for β^* as the optimum β. Our equation, (*i*) determines β^* exactly, in contrary to [1], where β^* is found numerically through an exhaustive search, and also, (*ii*) is consistent with the estimated equation for β^* in [1].

Finally, we illustrate numerically a comparison between our β^*, ℓ^* and upper bound with previous works.

Paper Organization

This paper has 4 sections. Section II, includes main results, an upper bound for $I_{q,\beta}(X;Y)$ is found and by maximizing the upper bound over all βs, we find the optimum β as β^* for each given SNR. In section III, we plot β^*, ℓ^* and the obtained upper bound versus SNR, discuss on the figures and compare them with previous works. The paper concludes in section IV.

2. Main Results

In this section, first we determine an upper bound for $I_{q,\beta}(X;W)$ in (11). Then we maximize the upper bound over all βs and find β* at each SNR as follows,

$$I_{q,\beta}(X;W) = -\int_{-\infty}^{\infty} \sum_{k=0}^{m} \frac{1}{1+\beta\frac{P}{\sigma}} \left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^k \frac{1}{\sqrt{2\pi\beta^2}}\, e^{-\frac{(w-k)^2}{2\beta^2}} .$$

$$\log_2 \sum_{i=0}^{m} \frac{1}{1+\beta\frac{P}{\sigma}} \left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^i \frac{1}{\sqrt{2\pi\beta^2}}\, e^{-\frac{(w-i)^2}{2\beta^2}} dw - \frac{1}{2}\log_2(2\pi e\beta^2) =$$

$$-\int_{-\infty}^{\infty} \sum_{k=0}^{m} \frac{1}{1+\beta\frac{P}{\sigma}} \left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^k \frac{1}{\sqrt{2\pi\beta^2}}\, e^{-\frac{(w-k)^2}{2\beta^2}}$$

$$.\left[\log_2\left(\frac{1}{1+\beta\frac{P}{\sigma}} . \frac{1}{\sqrt{2\pi\beta^2}}\right) + \log_2\left(\sum_{i=0}^{m} \left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^i e^{-\frac{(w-i)^2}{2\beta^2}}\right)\right]$$

$$dw - \frac{1}{2}\log_2(2\pi e\beta^2) \quad (13)$$

In (13), the term $\left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^i e^{-\frac{(w-i)^2}{2\beta^2}}$ is positive and less than 1. On the other hand we have [8],

$0 \le u_i \le 1 \rightarrow \sum u_i \ge \prod u_i \rightarrow -\log \sum u_i \le -\log\prod u_i$

$\rightarrow -\log \sum u_i \le -\sum \log u_i,$

using the inequality above, it can be written;

$$-\log_2\left[\sum_{i=0}^{m} \left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^i e^{-\frac{(w-i)^2}{2\beta^2}}\right] \le$$

$$-\sum_{i=0}^{m} \log_2 \left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^i e^{-\frac{(w-i)^2}{2\beta^2}} =$$

$$\underbrace{-\sum_{i=0}^{m} \log_2 \left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)^i}_{A} + \underbrace{\sum_{i=0}^{m} \frac{(w-i)^2}{2\beta^2}}_{B} . \log_2 e, \quad (14)$$

$$A = -\frac{m(m+1)}{2}\log_2\left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)$$

$$B = \frac{m+1}{2\beta^2} w^2 - \frac{m(m+1)}{2\beta^2} w + \frac{m(m+1)(2m+1)}{12\beta^2}$$

$$\int_{-\infty}^{\infty} f_W(w)\, dw = 1 \quad (15)$$

$$\int_{-\infty}^{\infty} w f_W(w) = E(w) = k \quad (16)$$

$$\int_{-\infty}^{\infty} w^2 f_W(w)\, dw = E(w^2) = \beta^2 + (E(w))^2 = \beta^2 + k^2, \quad (17)$$

in view of (14), (15), (16) and (17) the following upper bound is obtained,

$$I_{q,\beta}(X;W) \le -\log_2\left(\frac{1}{1+\beta\frac{P}{\sigma}}\cdot\frac{1}{\sqrt{2\pi\beta^2}}\right)-\frac{m^2+m}{2}\log_2\left(\frac{\beta\frac{P}{\sigma}}{1+\beta\frac{P}{\sigma}}\right)+\left(\frac{6(m+1)(\beta^2+k^2)-6(m^2+m)k+2m^3+3m^2+m}{12\beta^2}\right).\log_2 e-\frac{1}{2}\log_2(2\pi e\beta^2)=CUB. \quad (18)$$

2.1. Determining β^*

We should determine βs such that the upper bound in (18) as CUB becomes maximum for each P/σ,

$$\frac{\partial C_{UB}}{\partial\beta}=0\rightarrow$$

$$12\frac{P}{\sigma}\beta^3-6(m^2+m)\beta^2-\left(12(m+1)k^2-12(m^2+m)k+(4m^3+6m^2+2m)\right)\frac{P}{\sigma}\beta-12(m+1)k^2+12(m^2+m)k-4m^3-6m^2-2m=0, \quad (19)$$

we found a third order equation for β*. Solving equation (19) analytically we obtain three solutions β^*_1, β^*_2 and β^*_3 for β*,

$$\beta^*_1=\frac{d_2}{d_1}+d_1+\frac{d_3}{18P} \quad (20)$$

$$\beta^*_2=\frac{d_3}{18P}-\frac{d_1}{2}-\frac{d_2}{2d_1}-\frac{\sqrt{3}\left(\frac{d_2}{d_1}-d_1\right)j}{2}$$

$$\beta^*_3=\frac{d_3}{18P}-\frac{d_1}{2}-\frac{d_2}{2d_1}+\frac{\sqrt{3}\left(\frac{d_2}{d_1}-d_1\right)j}{2}$$

where,

$$d_1=\left(\sqrt{\left(\frac{6k^2m+6k^2-6km^2-6km+2m^3+3m^2+m}{12P}+\frac{d_3^3}{5832P^3}+\frac{d_3d_4}{216P^2}\right)^2-d_2^3}+\frac{6k^2m+6k^2-6km^2-6km+2m^3+3m^2+m}{12P}+\frac{d_3^3}{5832P^3}+\frac{d_3d_4}{216P^2}\right)^{1/3}$$

$$d_2=\frac{d_4}{18P}+\frac{d_3^2}{324P^2}$$

$$d_3=3m^2+3m$$

$$d_4=6Pk^2m+6Pk^2-6Pkm^2-6Pkm+2Pm^3+3Pm^2+Pm,$$

two of the solutions are complex and one of them is real. Note that for each P/σ the real solution i.e. β^*_1, is considered for β*. If we put m = 1 i.e. k = 0,1 (two mass points) in equation (19) we obtain,

$$12\frac{P}{\sigma}\beta^3-12\beta^2+12\frac{P}{\sigma}\beta-12=0\rightarrow$$

$$(\beta^2+1)\left(\frac{P}{\sigma}\beta-1\right)=0\rightarrow\beta^*=\frac{\sigma}{P}=\frac{1}{SNR}, \quad (21)$$

as we see, β* is a function of P/σ and this confirms the estimated β in [1] that is,

$$\tilde{\beta}=\frac{1}{c_1}\left(\frac{P}{\sigma}\right)^{-c_2-1}, \quad (22)$$

where $c_1 = 3.08$, $c_2 = -1.06$.

β* in (21) has an inverse relation with SNR, in contrary to the estimated β* in (22) that has a direct relation with SNR.

3. Numerical Results

In this section we plot β*, ℓ* and the upper bound that we have found versus SNR, discuss on the figures, and compare them with previous works.

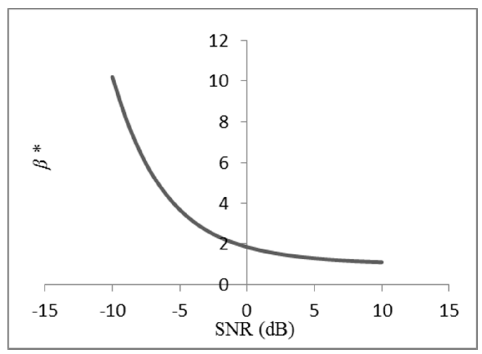

Fig. 1: The optimum β versus SNR.

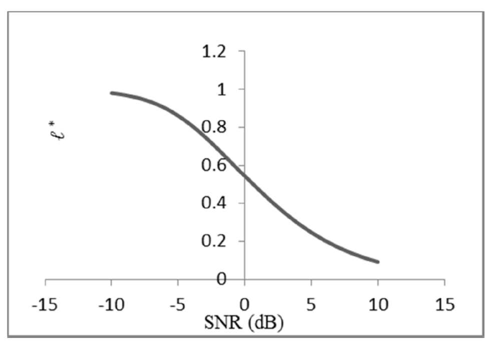

Fig. 2: The corresponding mass point spacing ℓ for P = 1.*

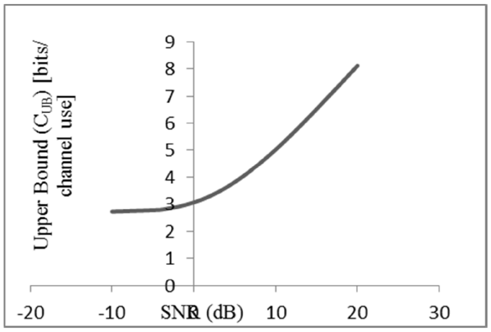

Fig.3: The obtained upper bound CUB versus SNR.

3.1. Discussions on the Figures and Comparisons with Previous Works

(i) Fig. 1 shows that our β*s are decreasing with SNR and it is in contrast with [1] that β*s are increasing with SNR.

(ii) The maximization in [1] is carried out by discretizing the range β ∈ (0,1] with an increment of Δβ = 5 × 10-4. For each entry in the discrete set, the mutual information is computed numerically and an exhaustive search is executed to find β*. But we have found β* through equation (19) exactly.

(iii) The optimum mass point spacing is also presented in Fig. 2 for average optical power P = 1. It is obvious that the mass point spacing should decrease with SNR as we have seen before in [1,2,3,5,6] and Fig. 2 confirms this fact too. ℓ* is approximated as a linear function of SNR in logarithmic domain in [1], but our ℓ* is linear just in the range of SNRs (−5 to 5 dB) and out of this range it will be nonlinear.

(iv) Fig. 3 shows the obtained upper bound versus SNR. At low SNRs there is a great gap between our upper bound and the upper bound in [1] as CU. But at high SNRs our upper bound becomes better, and at very high SNRs e.g. SNR > 17 dB, our upper bound becomes tighter than CU. It means that our upper bound has faster convergence behavior than CU at very high SNRs.

(v) In comparison with the upper bound obtained in [9] i.e. MU, our upper bound is tighter than MU at SNRs > 15 dB. However our upper bound has a simple analytic form.

(vi) The upper bound in [10] i.e. LMWU which is the modified version of MU is tighter than our upper bound at both low and high SNRs but LMWU is computed through a complicated expression and it is not clear how the parameters in the expression are chosen, in contrary to our upper bound that has a simple analytic form.

4. Conclusion

In this paper, we determined a new upper bound for the capacity of FSO channels by using a simple mathematical inequality and the FH work. Then by maximizing the upper bound over all the input parameters (βs) we found a third order equation for β* as the optimum β. Our equation (i) determines β* exactly, in contrary to the FH work where β* is found numerically, and also, (ii) is consistent with the estimated equation for β* in the FH work. Our upper bound is shown to be tighter than the proposed upper bound in the FH work that is found through sphere packing argument at very high SNRs. Our upper bound is tighter than the upper bound MU at SNRs > 15 dB and has a simple analytic form. The upper bound LMWU which is the modified version of MU is tighter than our upper bound at both low and high SNRs, but LMWU is computed through a complicated expression and it is not clear how the parameters in the expression are chosen, in contrary to our upper bound that has a simple analytic form.

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